adjusted for the additional data, the correct formulation compared very well with the raw bidirectional reflectance data.

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Note on Rectangular Finite Elements with In-Plane Forces

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QUADRILATERAL elements with linear boundaries are frequently used in solving two-dimensional elasticity problems by the finite element method. Although it is possible to generate element matrices with progressively higher-order approximation, the lower-order models are widely used in practice. For a four-node quadrilateral, fairly simple element matrices are commonly derived utilizing any of the following bases: 1) A composite of four constant strain triangles with static condensation of the centroidal node.[2,4] 2) Bilinear shape

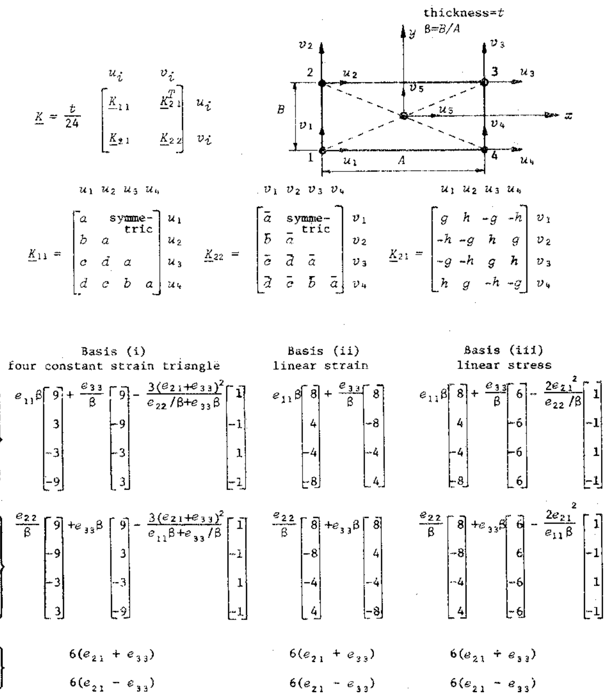

Fig. 1 Stiffness coefficients for a rectangle element.

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functions in the isoparametric coordinates which leads to linearly varying strain state for rectangular or parallelogram elements.[1,3,5,6] 3) As in Basis 2) with additional parabolic modes associated with free parameters which are finally reduced out.[5]

Under Bases 1) and 2), inter-element compatibility of displacement is maintained, whereas under Basis 3) it is violated. In everyday usage of finite element programs, rectangular elements (Fig. 1) are frequently used, and it is the principal objective here to point out certain special characteristics of the element matrices for such elements.

Using fairly simple algebra, explicit expressions for the reduced stiffness matrix under Basis 1) could be obtained as shown in Turner et al.[4] Under Basis 2), the stiffness coefficients are already available.[1] The modifications needed due to the addition of the parabolic modes in Basis 3) could also be worked out explicitly with some effort. Assuming zero reference temperature and an orthotropic stress-strain relation expressed by Eq. (1), the needed expressions for the stiffness coefficients are summarized in Fig. 1.

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}e_{11} & e_{21} & 0\\ e_{21} & e_{22} & 0\\ 0 & 0 & e_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{bmatrix}-T\begin{bmatrix}\gamma_{11}\\ \gamma_{22}\\ 0\end{bmatrix} \tag{1}$$

By taking the difference between the expressions shown in Fig. 1, the following interesting relations between the matrices obtained under different bases [indicated as superscripts in Eq. (2)] could be written.

$$\begin{aligned}
\mathbf{K}_{11}^{(i)}-\mathbf{K}_{11}^{(ii)} &= [e_{11}\beta+e_{33}/\beta-3(e_{21}+e_{33})^2/(e_{22}/\beta+e_{33}\beta)]\mathbf{N}\\
\mathbf{K}_{22}^{(i)}-\mathbf{K}_{22}^{(ii)} &= [e_{22}/\beta+e_{33}\beta-3(e_{21}+e_{33})^2/(e_{11}\beta+e_{33}/\beta)]\mathbf{N}\\
\mathbf{K}_{11}^{(ii)}-\mathbf{K}_{11}^{(iii)} &= (2e_{33}/\beta+2e_{21}{}^2/e_{22}/\beta)\mathbf{N}\\
\mathbf{K}_{22}^{(ii)}-\mathbf{K}_{22}^{(iii)} &= (2e_{33}\beta+2e_{21}{}^2/e_{11}\beta)\mathbf{N}
\end{aligned} \tag{2}$$

where

$$\mathbf{N}=\begin{bmatrix}1 & \text{symmetric} & & \\ -1 & 1 & & \\ 1 & -1 & 1 & \\ -1 & 1 & -1 & 1\end{bmatrix}$$

and the cross coefficient matrix $\mathbf{K}_{21}$ is the same under all bases, independent of the aspect ratio β, and is purely a function of the elasticity constants e_{21} and e_{33}.

Assuming a constant temperature T within an element, the thermal load vector $\mathbf{P}$ for an element could be expressed as

$$\mathbf{P}=\frac{t\cdot T}{2}[\overset{u_1}{-\gamma_{11}A}\quad \overset{u_2}{-\gamma_{11}A}\quad \overset{u_3}{\gamma_{11}A}\quad \overset{u_4}{\gamma_{11}A}\quad \overset{v_1}{-\gamma_{22}B}\quad \overset{v_2}{\gamma_{22}B}\quad \overset{v_3}{\gamma_{22}B}\quad \overset{v_4}{-\gamma_{22}B}] \tag{3}$$

and is independent of the bases considered here. It may be mentioned here that the reduced stiffness matrix derived under Basis 3) or the one derived in Przemieniecki[3] and Turner et al.[4] using linear-stress assumption leads to identical results for rectangular elements.

The knowledge of the structural similarities between the different matrices as outlined here may be fruitfully utilized in a programing code. Without much effort, it would be possible to incorporate the different formulation bases in a single package, offering a variety in the choice of the elements. For a certain class of problems permitting a regular element mesh, the full power and arbitrariness of the finite element method is not needed. With a suitable node numbering system and using the coefficients in Fig. 1, the equations for the global stiffness matrix could be written explicitly as a set of recursive difference equations, much like the way finite difference operator equations are written. Analytical techniques for solving recursive difference equations are already available. These could be fruitfully utilized in making parametric investigations of some problems, which, at present, require numerous computer runs using a general-purpose finite element code.

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Turbulence-Model Transition Predictions

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I. Introduction

ALTHOUGH analytical tools for predicting transition from laminar to turbulent flow have improved significantly in recent years, transition remains one of the least understood phenomena of fluid mechanics. Classical linear stability analysis has been used extensively. While some insight into the transition phenomenon has attended this work, predictions often differ from experimental observations. Furthermore, linear analysis determines stability of a flow to infinitesimal disturbances only and is inapplicable when initial flow perturbations are of finite amplitude. Finally, in a linear stability analysis, complicating effects such as wall roughness, wall cooling, mass transfer, and freestream turbulence level and scale considerably increase the approach's mathematical complexity.

An alternative analytical method for transition prediction exists which in principle a) is applicable to arbitrary amplitude disturbances and b) in a simple and natural way can account for all of the complicating effects cited above. This method uses the Reynolds-averaged equations of motion subject to a set of closure hypotheses suitable for accurate computation through transition. Recent progress with phenomenological-turbulence-model equations indicates that this approach is sensible, i.e., that adequate closure approximations can indeed be determined. Using turbulence-model equations in which the Reynolds stresses depend upon flow history, Donaldson,[1] Jones and Launder,[2] and Wilcox[3] have shown that such sets of turbulence-model equations accurately predict abrupt transition from laminar to turbulent flow for constant-pressure boundary layers.

This Note presents recent results of turbulence-model transition research based on the Saffman turbulence model.[4] As discussed in Sec. II, the turbulence model has been modified to improve computational accuracy for transition predictions; then effects of freestream turbulence intensity and scale on model-predicted transition for an incompressible flat-plate boundary layer (FPBL) have been analyzed using the modified equations. The model also has been used to predict the minimum suction required to prevent FPBL transition and to make estimates of transition Reynolds number in channel and pipe flow.

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II. Formulation

The Saffman turbulence model assumes Reynolds stress, $\langle -u'v' \rangle$, is proportional to the mean velocity gradient, $\partial u/\partial y$, i.e.,

$$\langle -u'v' \rangle = \varepsilon\, \partial u/\partial y \tag{1}$$

where ε is the eddy viscosity. The eddy viscosity is postulated to be the ratio of the turbulent energy e and a turbulent pseudovorticity or dissipation rate ω, so that

$$\varepsilon = e/\omega \tag{2}$$

For incompressible boundary-layer flows, the "turbulence densities" e and ω satisfy the following nonlinear diffusion equations:

$$u\,\partial e/\partial x + v\,\partial e/\partial y = [\alpha^*|\partial u/\partial y| - \beta^*\omega]e + (\partial/\partial y)[(v+\sigma^*\varepsilon)\partial e/\partial y] \tag{3}$$

$$u\,\partial\omega^2/\partial x + v\,\partial\omega^2/\partial y = [\alpha|\partial u/\partial y| - \beta\omega]\omega^2 + (\partial/\partial y)[(v+\sigma\varepsilon)\partial\omega^2/\partial y] \tag{4}$$

where x and y denote distance parallel to and normal to the surface, u and v are velocity components in the x and y directions, and v is kinematic viscosity. The six parameters α, α^*, β, β^*, σ, σ^* are regarded as universal constants for fully developed turbulent flows, and their values have been established by general arguments based on well-documented experimental observations for such flows.[4]

The Saffman turbulence model has been incorporated in a boundary-layer program developed at the NASA Langley Research Center[5]; the modified program is known as EDDYBL. In using EDDYBL to make FPBL transition predictions, turbulent energy and pseudovorticity are held constant at the boundary-layer edge. Turbulent energy is set to zero throughout the boundary layer at a point near the plate leading edge. We then march in the streamwise direction and observe behavior of e. Some entrainment of e into the boundary layer initially occurs; however little or no turbulent-energy amplification occurs for a plate-length Reynolds number below a critical value $Re_x{}^t$, signifying existence of laminar flow. Then, when $Re_x \sim Re_x{}^t$, an abrupt increase in e is observed, followed by an asymptote to a value characteristic of fully developed turbulent flow. The transitional regime is readily identified as the range over which e increases from its initially low level to its much higher value in the turbulent regime. The transitional regime can also be identified from the numerical data by locating abrupt changes in quantities such as momentum thickness, shape factor, and skin friction.

Figure 1 shows computed skin friction c_f as a function of Re_x for an incompressible FPBL. The freestream value of e is $10^{-9}U^2$, where U is freestream velocity. As shown in the figure, the predicted transition begins at $Re_x = 4 \times 10^4$ and ends at

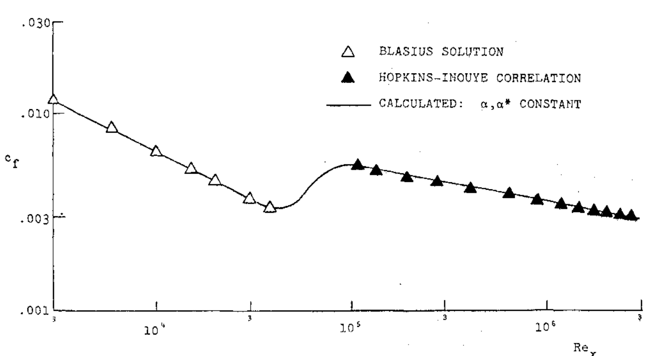

Fig. 1 Skin friction as a function of plate-length Reynolds number for an incompressible FPBL.